# QUASIPARTICLE TIME BLOCKING APPROXIMATION WITHIN THE FRAMEWORK OF GENERALIZED GREEN FUNCTION FORMALISM

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## Abstract

The problem of the microscopic description of excited states of the even-even open-shell atomic nuclei is considered. A model is formulated which allows one to go beyond the quasiparticle random phase approximation. The physical content of the model is determined by the quasiparticle time blocking approximation (QTBA) which enables one to include contributions of the two-quasiparticle and the two-phonon configurations, while excluding (blocking) more complicated intermediate states. In addition, the QTBA ensures consistent treatment of ground state correlations in the Fermi systems with pairing. The model is based on the generalized Green function formalism (GGFF) in which the normal and the anomalous Green functions are treated in a unified way in terms of the components of generalized Green functions in a doubled space. Modification of the GGFF is considered in the case when the many-body nuclear Hamiltonian contains two-, three-, and other many-particle effective forces.

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# 1 INTRODUCTION

One of the most widely used approaches which are applied to the description of excitations of the even-even atomic nuclei is the random phase approximation (RPA, see, e.g., Ref. [1]). Within this approximation the nuclear excitations are treated as the one-phonon states which are superpositions of the one-particle-one-hole (1p1h) configurations. However, this approach is applicable only in the case when pairing correlations are not essential, i.e., strictly speaking, only for magic nuclei. Generalization of the RPA taking into account pairing correlations explicitly is the quasiparticle RPA (QRPA), where the excited states (phonons) are expanded in the two-quasiparticle (2q) configurations. Thereby the QRPA extends the range of the RPA to the open-shell (non-magic) nuclei. Nevertheless, despite the significant success of both the RPA and the QRPA, there are several reasons for development of the models going beyond these approximations.

First of all, description of the nuclear excitations in terms of the one-phonon wave functions is justified only for low-lying states. At higher excitation energies, fragmentation of the one-phonon states becomes important. This means that in addition to the 1p1h or 2q configurations the more complex configurations should be incorporated (see Ref. [2]). The role of the effects related to the complex configurations is well manifested, for example, in the theory of giant multipole resonances (GMRs). It is well known (see, e.g., Ref. [3]) that RPA and QRPA enable one to describe the centroid energies and total strengths of the GMRs. However, both models fail to reproduce the total widths of the resonances and their fine structure. The reason is that these characteristics of the GMRs are significantly affected by the complex (mainly 2p2h or 4q) configurations which form the spreading width of the resonance.

Another direction of developing a nuclear structure theory is associated with the models in which ground state correlations (GSC) beyond the RPA and QRPA GSC are taken into account (see Refs. [3, 4, 5, 6]). It has been shown that the GSC caused by complex configurations play an important role in the theoretical description of the experimental data. In what follows we will refer to this type of GSC as the GSC2 in order to distinguish them from the GSC1 included in the RPA and QRPA. In some cases, the GSC2 can strongly affect the transition strengths and can even lead to the appearance of new transitions which are absent in the calculations including complex configurations in the excited states only, i.e. in the calculations neglecting this type of GSC and using a restricted basis (see Ref. [3]).

A variety of models have been developed to study the effects of complex configurations on the structure of excited states of the even-even atomic nuclei (in addition to the aforementioned papers see also Refs. [7, 8, 9, 10, 11, 12, 13] and references therein). Nevertheless, until recently (see Refs. [12, 13]), the quasiparticle-phonon model (QPM, Ref. [2]) developed by Soloviev and co-workers was the only working approach which consistently treats the complex configurations and the pairing correlations on an equal footing. It is no surprising that comprehensive studies of the excitations of the open-shell nuclei taking into account complex (mainly two-phonon) configurations at the microscopic level have been carried out only within the QPM. In view of this the development of other approaches in this direction is particularly important.

The principal goal of the present paper is to generalize the model of Ref. [10] by including the pairing correlations. This model was developed to describe the excited states of the even-even doubly magic nuclei taking into account 2p2h (more precisely, 1p1h⊗phonon) configurations. The model is based on the Green function (GF) formalism. The GSC2 were completely included within the model approximations. In the framework of this model the calculations of the GMRs in magic stable and unstable nuclei have been performed. Some of the results are presented in Refs. [3, 10]. In these calculations reasonable agreement with the experimental data for the integral characteristics of GMRs, including the total resonance widths, has been obtained. Thus, one can expect that the extension of the model [10] to the open-shell nuclei will also give reasonable results.

The second goal of the paper is to provide a modification of the GF formalism for the Fermi systems with pairing which is most suitable for solving the problem under consideration. To this aim, the generalized Green function formalism (GGFF) is presented, in which the normal and the anomalous GFs are treated in a unified way in terms of the components of generalized GFs in a doubled space. Modification of the GGFF is considered in the case when the many-body nuclear Hamiltonian contains two-, three-, and other many-particle effective forces.

The paper is divided into two main parts. The first part (Sec. 2) reviews the basic formulas and equations of the GGFF. The single-quasiparticle basis functions, which provide suitable representation of the model equations, are introduced. The second part (Sec. 3) contains the formulation of the model in which pairing correlations, 2q, 2q⊗phonon, and two-phonon configurations are included. The model is analyzed within sum rule approach. The conclusions are given in the last section.

## 2  GENERALIZED GREEN FUNCTION FORMALISM

### 2.1  Basic definitions

Let $a^\dagger(x)$ and $a(x)$ be creation and annihilation operators of particles (free fermions) in the coordinate representation of the usual single-particle space. Here symbol $x = \{\boldsymbol{r}, \sigma, \tau\}$ includes the spatial coordinate $\boldsymbol{r}$, the spin $\sigma$, and the isospin $\tau$ variables. Considering the Fermi systems with pairing correlations it is convenient to pass from this single-particle space spanned by the coordinates $x$ to the extended (doubled) space spanned by the coordinates $y = \{x, \chi\}$, where $\chi = \pm 1$ is an additional index introduced for denoting the different components of the single-particle functions in the extended space (see Refs. [1, 14] for details). Let us define the operators $b(y) = b(x, \chi)$ by the relations

$$b(x, +) = a(x), \qquad b(x, -) = a^\dagger(x). \tag{2.1}$$

From this it follows that $b^\dagger(y) = b(\bar{y})$ where $\bar{y} = \{x, -\chi\}$. The Heisenberg representation of the $b$-operators (in units where Planck's constant $\hbar = 1$) reads:

$$\Psi(z) = e^{iHt}\, b(y)\, e^{-iHt}. \tag{2.2}$$

Here and in the following $z = \{t, y\}$, $t$ is the time variable, $H$ is a many-body Hamiltonian of an interacting fermion system. Obviously, these $\Psi$-operators possess the property:

$$\Psi^\dagger(z) = \Psi(\bar{z}), \tag{2.3}$$

where $\bar{z} = \{t, \bar{y}\}$.

We will assume that the motion in the fermion system is determined by the nonrelativistic Hamiltonian $H$ of the form

$$H = H^0 + V, \tag{2.4}$$

where $H^0$ is a single-particle Hamiltonian including the external anomalous pair potentials:

$$\begin{aligned}
H^0 &= \int dx_1\, dx_1' \left( h^0(x_1, x_1')\, a^\dagger(x_1)\, a(x_1') \right. \\
&\quad \left. + \tfrac{1}{2}\, \Delta^0(x_1, x_1')\, a^\dagger(x_1)\, a^\dagger(x_1') - \tfrac{1}{2}\, \Delta^{0*}(x_1, x_1')\, a(x_1)\, a(x_1') \right),
\end{aligned} \tag{2.5}$$

$V$ is an interaction including two-, three-, and other many-particle effective forces:

$$V = \sum_{k=2}^{K} V^{(k)}, \tag{2.6}$$

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$$V^{(k)} = \frac{1}{k!} \int dx_1 \cdot \ldots \cdot dx_k \, dx_1' \cdot \ldots \cdot dx_k' \, v^{(k)}(x_1, \ldots, x_k \,;\, x_1', \ldots, x_k')$$
$$\times \, a^\dagger(x_1) \cdot \ldots \cdot a^\dagger(x_k) \, a(x_k') \cdot \ldots \cdot a(x_1') \,. \tag{2.7}$$

Here and in the following $\int dx$ means the space integral over $\boldsymbol{r}$ and the sum over $\sigma$ and $\tau$ indices. Analogously, in the following $\int dy$ will denote $\int dx$ and the sum over $\chi$, $\int dz$ will denote $\int dt\, dy$. In case of the exact nuclear Hamiltonian we have:

$$h^0(x, x') = -\left(\frac{\boldsymbol{\nabla}_{\mathbf{r}}^2}{2m} + \mu_\tau\right) \delta(x, x') \,, \qquad \Delta^0(x, x') = 0 \,, \tag{2.8}$$

where $\delta(x, x') = \delta(\boldsymbol{r} - \boldsymbol{r}') \, \delta_{\sigma,\sigma'} \, \delta_{\tau,\tau'}$, $\mu_\tau$ is the chemical potential for the nucleons with the isospin projection $\tau$ which is introduced to simplify the following equations. Notice that the Hamiltonian $H^0$ can be formally rewritten in terms of the $b$-operators as

$$H^0 = \frac{1}{2} \int dy\, dy' \, \mathcal{H}^0(y, y') \, b^\dagger(y) \, b(y') \, + \, \epsilon_0 \,, \tag{2.9}$$

where

$$\left.\begin{array}{ll} \mathcal{H}^0(x, +, \, x', +) = h^0(x, x') \,, & \mathcal{H}^0(x, +, \, x', -) = \Delta^0(x, x') \,, \\[2mm] \mathcal{H}^0(x, -, \, x', +) = -\Delta^{0*}(x, x') \,, & \mathcal{H}^0(x, -, \, x', -) = -h^{0*}(x, x') \,, \end{array}\right\} \tag{2.10}$$

$$\epsilon_0 = \frac{1}{2} \int dx \, h^0(x, x) \,. \tag{2.11}$$

Let $|\,0\,\rangle$ be the wave function of the ground state of the interacting fermion system. If the Hamiltonian $H$ does not contain the external anomalous pair potentials $\Delta^0$ [e.g., if Eqs. (2.8) are fulfilled], the number of particles is conserved, and $|\,0\,\rangle$ is an eigenfunction of the particle-number operator. However, in general case we shall not suppose that $\Delta^0 = 0$ in Eq. (2.5), i.e. we shall not suppose that the condition of the particle-number conservation is fulfilled for $|\,0\,\rangle$. Let us define the $k$-particle generalized Green function (GF) in the time representation by the formula:

$$G^{(k)}(z_1, \ldots, z_k \,;\, z_1', \ldots, z_k') = i^{-k} \, \langle\, 0\, |\, \mathrm{T} \, \Psi(z_1) \cdot \ldots \cdot \Psi(z_k) \, \Psi^\dagger(z_k') \cdot \ldots \cdot \Psi^\dagger(z_1') \,|\, 0\,\rangle, \tag{2.12}$$

where T is the time-ordering operator. In particular, for the single-particle GF we have:

$$G(z, z') \equiv G^{(1)}(z \,;\, z') = -i \, \langle\, 0\, |\, \mathrm{T} \, \Psi(z) \, \Psi^\dagger(z') \,|\, 0\,\rangle \,. \tag{2.13}$$

The property

$$G(z, z') = -G(\bar{z}', \bar{z}) \tag{2.14}$$

follows from Eqs. (2.3) and (2.13). It can be seen from the definitions (2.1) and (2.2) that the normal and the anomalous GFs are the components of the generalized GFs $G^{(k)}$ corresponding to the different values of the $\chi$-indices.

## 2.2 Equations of motion for the Green functions

In case of the Fermi systems with pairing, the equations of motion for the many-particle GFs can be obtained with the help of the same technique based on the generating functionals depending on the auxiliary source fields which is frequently used for the Fermi systems without pairing correlations (see, e.g., Ref [15]). Let us define the generating functional $W$ depending on the source field $\xi$ as

$$W[\xi] = \ln\langle\, 0\,|\,T\,U\,|\,0\,\rangle\,, \tag{2.15}$$

where

$$U = \exp\left(i\int dz\,dz'\,\xi(z,z')\,\Psi^\dagger(z)\,\Psi(z')\right). \tag{2.16}$$

It follows from Eq. (2.3) that one can consider the equality

$$\xi(z,z') = -\xi(\bar{z}',\bar{z}) \tag{2.17}$$

to be fulfilled. Let us introduce the GFs with a source field $\xi$:

$$G_\xi^{(k)}(z_1,\ldots,z_k;\,z_1',\ldots,z_k') = i^{-k}\,\frac{\langle\,0\,|\,T\,U\Psi(z_1)\cdot\ldots\cdot\Psi(z_k)\,\Psi^\dagger(z_k')\cdot\ldots\cdot\Psi^\dagger(z_1')\,|\,0\,\rangle}{\langle\,0\,|\,T\,U\,|\,0\,\rangle}. \tag{2.18}$$

In particular, we have:

$$G_\xi(z,z') \equiv G_\xi^{(1)}(z;\,z') = -i\,\frac{\langle\,0\,|\,T\,U\Psi(z)\,\Psi^\dagger(z')\,|\,0\,\rangle}{\langle\,0\,|\,T\,U\,|\,0\,\rangle} = -G_\xi(\bar{z}',\bar{z})\,. \tag{2.19}$$

Obviously, $G_\xi^{(k)}$ coincides with $G^{(k)}$ defined by Eq. (2.12) at $\xi = 0$.

It is easy to see that the GFs $G_\xi^{(k)}$ can be obtained from the generating functional $W[\xi]$ by a successive differentiation with respect to $\xi$. In particular, we obtain:

$$G_\xi(z_1,z_2) = \frac{\delta W}{\delta\xi(z_2,z_1)}\,, \tag{2.20}$$

$$L_\xi(z_1,z_2;\,z_3,z_4) = \frac{\delta^2 W}{\delta\xi(z_1,z_2)\,\delta\xi(z_4,z_3)} = \frac{\delta G_\xi(z_2,z_1)}{\delta\xi(z_4,z_3)}\,, \tag{2.21}$$

where $L_\xi$ is the response function defined as

$$L_\xi(z_1,z_2;\,z_3,z_4) = G_\xi^{(2)}(z_2,z_3;\,z_1,z_4) - G_\xi(z_2,z_1)\,G_\xi(z_3,z_4) \tag{2.22}$$

(notice that this formula differs from the definition in Ref [15] by the permutation of the arguments of $L_\xi$).

The equation of motion for the single-particle GF is obtained by the differentiation of $G_\xi(z_1, z_2)$ with respect to time in analogy to the case of the Fermi systems without pairing correlations (see Ref [15]). It has the form

$$\left(G^0\right)^{-1}(z_1, z_2) + \xi(z_1, z_2) - \xi(\bar{z}_2, \bar{z}_1) = G_\xi^{-1}(z_1, z_2) + \Sigma_\xi(z_1, z_2), \qquad (2.23)$$

where

$$\left(G^0\right)^{-1}(z_1, z_2) = \left( i\, \delta(y_1, y_2)\, \frac{\partial}{\partial t_1} - \mathcal{H}^0(y_1, y_2) \right) \delta(t_1 - t_2), \qquad (2.24)$$

$\delta(y, y') = \delta_{\chi, \chi'}\, \delta(x, x')$, $\Sigma_\xi$ is the mass operator which is defined by the equations:

$$
\begin{aligned}
\int dz''\, \Sigma_\xi(z, z'')\, G_\xi(z'', z') \;=\;& \sum_{k=2}^{K} \frac{i^{1-k}}{k!\,(k-1)} \int dz_2' \cdot \ldots \cdot dz_k'\, dz_1'' \cdot \ldots \cdot dz_k'' \\
& \times \mathcal{W}^{(k)}(z, z_2', \ldots, z_k';\, z_1'', \ldots, z_k'') \\
& \times G_\xi^{(k)}(z_1'', \ldots, z_k'';\, z', z_2', \ldots, z_k'), \qquad (2.25)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}^{(k)}(z_1, \ldots, z_k;\, z_1', \ldots, z_k') \;=\;& \delta_{\chi_1, \chi_1'}\, \delta(t_1 - t_1' - \chi_1 \cdot 0) \\
& \times \left( \prod_{l=1}^{k-1} \delta_{\chi_l, \chi_{l+1}}\, \delta_{\chi_l', \chi_{l+1}'}\, \delta(t_l - t_{l+1})\, \delta(t_l' - t_{l+1}') \right) \\
& \times \big[\, \delta_{\chi_1, +1}\, w^{(k)}(x_1, \ldots, x_k;\, x_1', \ldots, x_k') \\
& + \delta_{\chi_1, -1}\, (-1)^k\, w^{(k)}(x_1', \ldots, x_k';\, x_1, \ldots, x_k)\,\big]. \qquad (2.26)
\end{aligned}
$$

In the Eq. (2.26), $w^{(k)}$ is the antisymmetrized matrix element of the $k$-particle interaction in the coordinate representation which is defined through the effective forces $v^{(k)}$ entering Eq. (2.7) and through the generalized antisymmetrized delta functions by the formulas:

$$
\begin{aligned}
w^{(k)}(x_1, \ldots, x_k;\, x_1', \ldots, x_k') \;=\;& \frac{1}{k!\,(k-2)!} \int dx_1'' \cdot \ldots \cdot dx_k''\, dx_1''' \cdot \ldots \cdot dx_k''' \\
& \times \delta\!\begin{pmatrix} x_1'', \ldots, x_k'' \\ x_1, \ldots, x_k \end{pmatrix} \delta\!\begin{pmatrix} x_1''', \ldots, x_k''' \\ x_1', \ldots, x_k' \end{pmatrix} \\
& \times v^{(k)}(x_1'', \ldots, x_k'';\, x_1''', \ldots, x_k'''), \qquad (2.27)
\end{aligned}
$$

where

$$
\delta\!\begin{pmatrix} x_1', \ldots, x_k' \\ x_1, \ldots, x_k \end{pmatrix} = \det \begin{pmatrix} \delta(x_1, x_1') & \ldots & \delta(x_1, x_k') \\ \ldots & \ldots & \ldots \\ \delta(x_k, x_1') & \ldots & \delta(x_k, x_k') \end{pmatrix}. \qquad (2.28)
$$

Notice that from Eqs. (2.10), (2.14), (2.23), and (2.24) it follows:

$$\Sigma_\xi(z, z') = -\Sigma_\xi(\bar{z}', \bar{z}). \qquad (2.29)$$

The symmetry property of $\mathcal{W}^{(k)}$ follows from its definition (2.26):

$$\mathcal{W}^{(k)}(z_1, \ldots, z_k; z_1', \ldots, z_k') = (-1)^k \, \mathcal{W}^{(k)}(\bar{z}_1', \ldots, \bar{z}_k'; \bar{z}_1, \ldots, \bar{z}_k) \,. \tag{2.30}$$

In order to obtain equations for the other (many-particle) GFs let us perform a change of the functional variable $\xi$ to the $G_\xi$ and consider a Legendre transformation of the functional $W[\xi]$:

$$\Gamma[G_\xi] = 2\,W[\xi] - \int dz_1 \, dz_2 \, [\, \xi(z_2, z_1) - \xi(\bar{z}_1, \bar{z}_2) \,] \, G_\xi(z_1, z_2) \,. \tag{2.31}$$

Using Eqs. (2.14) and (2.20) we obtain

$$\frac{\delta \Gamma}{\delta_- G_\xi(z_1, z_2)} = \xi(\bar{z}_1, \bar{z}_2) - \xi(z_2, z_1) \,. \tag{2.32}$$

The notation $\delta_-$ means that the variations of the GF $G_\xi$ conserve the property of antisymmetry (2.14). This condition should be taken into account since variations of $\xi$, which generate the variations of $G_\xi$, obviously do not lead to the violation of Eq. (2.14). Conservation of the property (2.14) in the variational procedure can be automatically ensured if the following substitution is performed in a $G_\xi$-dependent functional:

$$G_\xi(z, z') = \tfrac{1}{2}[\, G_\xi(z, z') - G_\xi(\bar{z}', \bar{z}) \,] \,. \tag{2.33}$$

In case of the vanishing source field the Eq. (2.32) leads to the stationarity condition:

$$\frac{\delta \Gamma}{\delta_- G(z_1, z_2)} = 0 \,. \tag{2.34}$$

Using, further, Eq. (2.23) we obtain from Eq. (2.32) the following relation

$$
\begin{aligned}
\frac{\delta \Sigma_\xi(z_2, z_1)}{\delta_- G_\xi(z_4, z_3)} &= \tfrac{1}{2} \left[\, G_\xi^{-1}(z_2, z_4) \, G_\xi^{-1}(z_3, z_1) - G_\xi^{-1}(\bar{z}_1, z_4) \, G_\xi^{-1}(z_3, \bar{z}_2) \,\right] \\
&\quad - \frac{\delta^2 \Gamma}{\delta_- G_\xi(z_1, z_2) \, \delta_- G_\xi(z_4, z_3)} \,.
\end{aligned}
\tag{2.35}
$$

Let us introduce an amplitude of the effective interaction $\mathcal{I}_\xi$ which includes irreducible amplitudes both in the particle-hole (ph), and in the particle-particle (pp) channels:

$$\mathcal{I}_\xi(z_1, z_2; z_3, z_4) = i\,\frac{\delta \Sigma_\xi(z_2, z_1)}{\delta_- G_\xi(z_4, z_3)} \,. \tag{2.36}$$

From Eqs. (2.14) and (2.29) we obtain:

$$\mathcal{I}_\xi(z_1, z_2; z_3, z_4) = -\mathcal{I}_\xi(\bar{z}_2, \bar{z}_1; z_3, z_4) = -\mathcal{I}_\xi(z_1, z_2; \bar{z}_4, \bar{z}_3) \,. \tag{2.37}$$

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In addition from Eq. (2.35) it follows that

$$\mathcal{I}_\xi(z_1, z_2; z_3, z_4) = \mathcal{I}_\xi(z_4, z_3; z_2, z_1). \tag{2.38}$$

Notice that the response function defined by Eq. (2.21) satisfies the analogous equalities

$$L_\xi(z_1, z_2; z_3, z_4) = -L_\xi(\bar{z}_2, \bar{z}_1; z_3, z_4) = -L_\xi(z_1, z_2; \bar{z}_4, \bar{z}_3), \tag{2.39}$$

$$L_\xi(z_1, z_2; z_3, z_4) = L_\xi(z_4, z_3; z_2, z_1). \tag{2.40}$$

Differentiating Eq. (2.23) with respect to $\xi$ and then using Eqs. (2.21) and (2.36), we obtain the Bethe-Salpeter equation (BSE) for the response function:

$$
\begin{aligned}
L_\xi(z_1, z_2; z_3, z_4) &= G_\xi(\bar{z}_4, z_1) G_\xi(z_2, \bar{z}_3) - G_\xi(z_3, z_1) G_\xi(z_2, z_4) \\
&\quad - i \int dz_5 \, dz_6 \, dz_7 \, dz_8 \, G_\xi(z_5, z_1) G_\xi(z_2, z_6) \\
&\quad \times \mathcal{I}_\xi(z_5, z_6; z_7, z_8) L_\xi(z_7, z_8; z_3, z_4).
\end{aligned}
\tag{2.41}
$$

The equations for the many-particle GFs $G_\xi^{(k)}$ with $k > 2$ are obtained by a differentiation of Eq. (2.18) with respect to $\xi$. Taking into account relation:

$$\frac{\delta}{\delta\xi(z_1, z_2)} = \int dz_3 \, dz_4 \, L_\xi(z_1, z_2; z_3, z_4) \frac{\delta}{\delta G_\xi(z_3, z_4)}, \tag{2.42}$$

which follows from Eq. (2.21), we come to the recurrence formula

$$
\begin{aligned}
G_\xi^{(k)}(z_1, \ldots, z_k; z_1', \ldots, z_k') &= \left[ G_\xi(z_1, z_1') + \int dz \, dz' \, L_\xi(z_1', z_1; z', z) \right. \\
&\quad \left. \times \frac{\delta}{\delta G_\xi(z', z)} \right] G_\xi^{(k-1)}(z_2, \ldots, z_k; z_2', \ldots, z_k').
\end{aligned}
\tag{2.43}
$$

Notice that in Eqs. (2.42) and (2.43) we have: $\delta_- G_\xi = \delta G_\xi$ owing to presence of the response function satisfying Eqs. (2.39).

The Eqs. (2.22), (2.25), (2.36), (2.41), and (2.43) form the closed system of the functional differential equations of the GGFF. An important feature of these equations is that they do not change their form when the many-particle forces are added to the two-particle interaction in the total Hamiltonian. The exception is Eq. (2.25) for the mass operator in which the many-particle forces enter in the explicit form. This result of the GGFF could be expected, but it needed to be proved. It allows to extend the standard GF methods developed for the Fermi systems with two-particle interaction to the systems interacting through the many-particle effective forces.

In the final equations we can set $\xi = 0$. So, in what follows we shall omit the $\xi$-indices of the functions implying the limit at $\xi = 0$ and coming back to the GFs without source field defined by Eq. (2.12). The independent functional variable is the single-particle GF $G$ satisfying the Dyson equation which follows from Eq. (2.23):

$$G(z_1, z_2) = G^0(z_1, z_2) + \int dz_3 \, dz_4 \, G^0(z_1, z_3) \, \Sigma(z_3, z_4) \, G(z_4, z_2) \, . \tag{2.44}$$

In case of the Fermi systems without pairing correlations, when the external anomalous pair potentials vanish in Eq. (2.5), the above equations can be reduced to ones for the normal components of GFs which satisfy the condition:

$$\sum_{n=1}^{k} (\chi_n - \chi'_n) \, G^{(k)}(z_1, \ldots, z_k \, ; \, z'_1, \ldots, z'_k) = 0 \, . \tag{2.45}$$

These reduced equations will coincide with ones obtained in Ref. [16]. On the other hand, if we restrict consideration of the Fermi systems with pairing to the case of two-particle interaction in the Hamiltonian (2.4), i.e. if we assume:

$$H = H^0 + V^{(2)} \, , \tag{2.46}$$

the Eqs. (2.22), (2.25), (2.36), and (2.41) will coincide up to the permutation of the arguments with the corresponding equations in Ref. [17].

## 2.3 Transformation of the basic equations

It is easy to see that any amplitude $\mathcal{I}$ possessing the properties (2.37) and (2.38) can be represented in the form

$$\mathcal{I}(z_1, z_2; z_3, z_4) = \tfrac{1}{2} \left[ \mathcal{U}(z_1, z_2; z_3, z_4) - \mathcal{U}(\bar{z}_2, \bar{z}_1; z_3, z_4) \right] , \tag{2.47}$$

where the amplitude $\mathcal{U}$ satisfies the equalities

$$\mathcal{U}(z_1, z_2; z_3, z_4) = \mathcal{U}(z_4, z_3; z_2, z_1) = \mathcal{U}(\bar{z}_2, \bar{z}_1; \bar{z}_4, \bar{z}_3) , \tag{2.48}$$

but generally $\mathcal{U}$ does not possess the property of antisymmetry (2.37). So in the following we shall refer to the $\mathcal{U}$ as the non-antisymmetric amplitude of the effective interaction. Obviously, the Eq. (2.47) does not define $\mathcal{U}$ uniquely. To specify the definition let us note that Eqs. (2.37) are fulfilled if the functional variable of $\Sigma_\xi$ in Eq. (2.36) is taken in the form

(2.33). However, one can formally define a functional derivative in which the substitution (2.33) is not supposed. Following this definition we set:

$$\mathcal{U}(z_1, z_2; z_3, z_4) = i \frac{\delta \Sigma(z_2, z_1)}{\delta G(z_4, z_3)}, \tag{2.49}$$

where the condition (2.14) is not supposed to be fulfilled under variations of $G$.

As an example, let us consider the first order in the two-particle interaction. From Eqs. (2.25), (2.22), and (2.41) we have:

$$\Sigma^{[1]}(z_1, z_2) = i \int dz_3 \, dz_4 \left[ \mathcal{W}^{(2)}(z_1, z_4; z_3, z_2) - \tfrac{1}{2} \mathcal{W}^{(2)}(z_1, \bar{z}_2; z_3, \bar{z}_4) \right] G(z_3, z_4). \tag{2.50}$$

$\Sigma^{[1]}$ is the generalized Hartree-Fock-Bogoliubov (HFB) contribution into the mass operator [the term *generalized* is used since the exact single-particle GF enters Eq. (2.50)]. Using, further, the definitions (2.49) and (2.36) we obtain from Eq. (2.50):

$$\mathcal{U}^{[1]}(z_1, z_2; z_3, z_4) = \mathcal{W}^{(2)}(z_2, z_3; z_1, z_4) + \tfrac{1}{2} \mathcal{W}^{(2)}(z_2, \bar{z}_1; z_4, \bar{z}_3), \tag{2.51}$$

$$\mathcal{I}^{[1]}(z_1, z_2; z_3, z_4) = \tfrac{1}{2} \left[ \mathcal{U}^{[1]}(z_1, z_2; z_3, z_4) - \mathcal{U}^{[1]}(\bar{z}_2, \bar{z}_1; z_3, z_4) \right]. \tag{2.52}$$

It can be seen that Eqs. (2.48) for the $\mathcal{U}^{[1]}$ follow from the antisymmetry of the $\mathcal{W}^{(k)}$ and from the property (2.30), while Eqs. (2.37) are not fulfilled.

For the further applications it is convenient to introduce a non-antisymmetric response function $R$ satisfying the following BSE:

$$\begin{aligned} R(z_1, z_2; z_3, z_4) &= R^0(z_1, z_2; z_3, z_4) + i \int dz_5 \, dz_6 \, dz_7 \, dz_8 \\ &\quad \times R^0(z_1, z_2; z_5, z_6) \mathcal{U}(z_5, z_6; z_7, z_8) R(z_7, z_8; z_3, z_4), \end{aligned} \tag{2.53}$$

where

$$R^0(z_1, z_2; z_3, z_4) = -G(z_3, z_1) G(z_2, z_4). \tag{2.54}$$

It is easy to show, first that the following equalities hold:

$$R(z_1, z_2; z_3, z_4) = R(z_4, z_3; z_2, z_1) = R(\bar{z}_2, \bar{z}_1; \bar{z}_4, \bar{z}_3), \tag{2.55}$$

while the Eqs. (2.39) do not hold for $R$. Second, the response function $L$, which satisfies Eq. (2.41), is expressed in terms of $R$ as:

$$L(z_1, z_2; z_3, z_4) = R(z_1, z_2; z_3, z_4) - R(\bar{z}_2, \bar{z}_1; z_3, z_4). \tag{2.56}$$

Thus, for the determination of the response function $L$ it is sufficient to solve Eq. (2.53) for the non-antisymmetric function $R$.

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Now, following the method described in Ref. [3], we represent the total mass operator $\Sigma$ and the total non-antisymmetric amplitude of the effective interaction $\mathcal{U}$ as a sum of two terms:

$$\Sigma = \tilde{\Sigma} + \Sigma^e , \qquad \mathcal{U} = \tilde{\mathcal{U}} + \mathcal{U}^e , \tag{2.57}$$

where

$$\tilde{\Sigma}(z_1, z_2) = \tilde{\Sigma}(y_1, y_2)\, \delta(t_1 - t_2) , \tag{2.58}$$

$$\tilde{\mathcal{U}}(z_1, z_2; z_3, z_4) = \tilde{\mathcal{U}}(y_1, y_2; y_3, y_4)\, \delta(t_1 - t_2)\, \delta(t_3 - t_4)\, \delta(t_1 - t_3) . \tag{2.59}$$

After transformation to the energy representation (see Sec. 2.4) the first terms in Eqs. (2.57), $\tilde{\Sigma}$ and $\tilde{\mathcal{U}}$, are found to be energy-independent. The term $\tilde{\Sigma}$ corresponds to the mean-field contribution into the mass operator including the pair potentials. The term $\tilde{\mathcal{U}}$ corresponds to the residual energy-independent interaction both in the ph, and in the pp channels. The second terms in Eqs. (2.57), $\Sigma^e$ and $\mathcal{U}^e$, have a strong energy dependence and represent dynamic contributions of complex configurations.

We stress that the quantities $\tilde{\Sigma}$ and $\tilde{\mathcal{U}}$ (and, consequently, $\Sigma^e$ and $\mathcal{U}^e$) are not defined rigorously by Eqs. (2.57). They will be specified in the following within the framework of the model to be considered. At the moment only the general properties, which are expressed by Eqs. (2.58) and (2.59), are important. Notice, however, that in particular case of the self-consistent HFB approximation restricted by the two-particle interaction we have: $\tilde{\Sigma} = \Sigma^{[1]}$, $\tilde{\mathcal{U}} = \mathcal{U}^{[1]}$, where the right-hand sides are defined by Eqs. (2.50) and (2.51) with the exact GF $G$ being replaced by the HFB GF $\tilde{G}$ which is the solution of Eq. (2.61) (see below).

Using decompositions (2.57) one can transform both of Eqs. (2.44) and (2.53) to the system of two equations. In the symbolic notations we have:

$$G = \tilde{G} + \tilde{G}\,\Sigma^e\,G , \tag{2.60}$$

$$\tilde{G} = G^0 + G^0\,\tilde{\Sigma}\,\tilde{G} , \tag{2.61}$$

$$R = R^e + i\,R^e\,\tilde{\mathcal{U}}\,R , \tag{2.62}$$

$$R^e = R^0 + i\,R^0\,\mathcal{U}^e\,R^e . \tag{2.63}$$

Proceeding by the same method as in Refs. [3, 10] the last equation can be brought to the following form:

$$R^e(z_1, z_2; z_3, z_4) = \tilde{R}^0(z_1, z_2; z_3, z_4) + i \int dz_5\, dz_6\, dz_7\, dz_8$$
$$\times\, \tilde{R}^0(z_1, z_2; z_5, z_6)\, \mathcal{W}^e(z_5, z_6; z_7, z_8)\, R^e(z_7, z_8; z_3, z_4) , \tag{2.64}$$

where

$$\tilde{R}^0(z_1, z_2; z_3, z_4) = -\tilde{G}(z_3, z_1)\,\tilde{G}(z_2, z_4)\,, \tag{2.65}$$

$$\mathcal{W}^e(z_1, z_2; z_3, z_4) = \mathcal{V}^e(z_1, z_2; z_3, z_4) - i\,\Sigma^e(z_3, z_1)\,\Sigma^e(z_2, z_4)\,, \tag{2.66}$$

$$\mathcal{V}^e(z_1, z_2; z_3, z_4) = \mathcal{U}^e(z_1, z_2; z_3, z_4)$$
$$+ i\,\Sigma^e(z_3, z_1)\,\tilde{G}^{-1}(z_2, z_4) + i\,\tilde{G}^{-1}(z_3, z_1)\,\Sigma^e(z_2, z_4)\,. \tag{2.67}$$

The Eq. (2.64) is the basic one for the building of model which will be considered in the second part of the paper.


## 2.4 Single-quasiparticle basis functions and the energy representation

For the following analysis it is required to introduce a set of basis functions $\{\psi_1(y)\}$ in the extended space defined previously in Sec. 2.1. The usual conditions of orthonormality and completeness are supposed to be fulfilled:

$$\int dy\,\psi_1^*(y)\,\psi_{1'}(y) = \delta_{1,1'}\,, \qquad \sum_1 \psi_1^*(y)\,\psi_1(y') = \delta(y, y')\,. \tag{2.68}$$

It is convenient to consider $\psi_1(y)$ to be the eigenfunctions of the operator:

$$\mathcal{H}(y, y') = \mathcal{H}^0(y, y') + \tilde{\Sigma}(y, y')\,, \tag{2.69}$$

where $\mathcal{H}^0$ defines the single-particle term of the total Hamiltonian according to Eqs. (2.9) and (2.10), $\tilde{\Sigma}$ is the mean-field contribution into the total mass operator in Eqs. (2.57). Thus, we assume the following equation to be fulfilled:

$$\int dy'\,\mathcal{H}(y, y')\,\psi_1(y') = E_1\psi_1(y)\,. \tag{2.70}$$

Since $\mathcal{H}$ possesses the same symmetry properties as the operators $\mathcal{H}^0$ and $\tilde{\Sigma}$, i.e.:

$$\mathcal{H}(y, y') = \mathcal{H}^*(y', y) = -\mathcal{H}(\bar{y}', \bar{y})\,, \tag{2.71}$$

it is not difficult to see that the complete set of the eigenfunctions of $\mathcal{H}$ is divided into two equal parts which are related by the operation of conjugation:

$$\psi_{\bar{1}}(y) = \psi_1^*(\bar{y})\,. \tag{2.72}$$

For the corresponding eigenvalues we have: $E_{\bar{1}} = -E_1$. So one can denote: $1 = \{\lambda_1, \eta_1\}$, $\bar{1} = \{\lambda_1, -\eta_1\}$, where $\lambda_1$ is the index of the usual single-particle configuration space (e.g.,

$\lambda = \{\tau_\lambda, n, l, j, m\}$ for the spherically symmetric system), $\eta_1 = \pm 1$ is the sign of the eigenvalue $E_1$:

$$E_1 = \eta_1 E_{\lambda_1}, \qquad E_{\lambda_1} = |E_1|. \tag{2.73}$$

In the representation of $\psi$-functions, the operators $b$ defined by Eq. (2.1) have the form

$$b_1 = \int dy\, \psi_1^*(y)\, b(y). \tag{2.74}$$

It is worth noting that the $b$-operators in this representation are simply related to the creation and annihilation operators of the quasiparticles $\alpha_\lambda^\dagger$ and $\alpha_\lambda$ which are usually introduced in the HFB theory. Namely, we have (see Ref. [14]): $b_{\lambda,+} = \alpha_\lambda$, $b_{\lambda,-} = \alpha_\lambda^\dagger$. So, in what follows we shall refer to the functions $\psi_1(y)$ as the single-quasiparticle functions.

A more detailed form of the functions $\psi_1(y)$ can be obtained making use of the Bloch-Messiah theorem, see Refs. [1, 18]. To formulate the result let us note that for time-reversal invariant fermion system, the set of the single-particle indices $\lambda$ can be divided into two subsets of conjugate indices $p$ and $\bar{p}$ which represent "paired" states, i.e.: $\{\lambda\} = \{p\} \cup \{\bar{p}\}$. In particular, for spherically symmetric system the conjugate indices are $\lambda = \{(\lambda), m\}$ and $\bar{\lambda} = \{(\lambda), -m\}$ where $(\lambda) = \{\tau_\lambda, n, l, j\}$. According to the Bloch-Messiah theorem the single-quasiparticle functions can be represented in the following form (see Ref. [14]):

$$\psi_{\lambda,+}(y) = \sum_{\lambda'} C_{\lambda'\lambda}\, \check{\psi}_{\lambda',+}(y), \qquad \psi_{\lambda,-}(y) = \sum_{\lambda'} C_{\lambda'\lambda}^*\, \check{\psi}_{\lambda',-}(y), \tag{2.75}$$

where $C_{\lambda'\lambda}$ is a unitary matrix and the functions $\check{\psi}_{\lambda,\eta}(y)$ have the form:

$$\left.\begin{array}{rclcrcl}
\check{\psi}_{p,+}(x,+) &=& u_p\, \varphi_p(x), & & \check{\psi}_{\bar{p},+}(x,+) &=& u_p\, \varphi_{\bar{p}}(x), \\
\check{\psi}_{p,+}(x,-) &=& -v_p\, \varphi_{\bar{p}}^*(x), & & \check{\psi}_{\bar{p},+}(x,-) &=& v_p\, \varphi_p^*(x), \\
\check{\psi}_{p,-}(x,+) &=& -v_p\, \varphi_{\bar{p}}(x), & & \check{\psi}_{\bar{p},-}(x,+) &=& v_p\, \varphi_p(x), \\
\check{\psi}_{p,-}(x,-) &=& u_p\, \varphi_p^*(x), & & \check{\psi}_{\bar{p},-}(x,-) &=& u_p\, \varphi_{\bar{p}}^*(x).
\end{array}\right\} \tag{2.76}$$

Here $\{\varphi_\lambda(x)\}$ is a complete set of orthonormal functions in the usual single-particle space, $v_\lambda$ and $u_\lambda$ are real non-negative numbers which satisfy the following conditions: $u_\lambda = \sqrt{1 - v_\lambda^2}$, $v_p = v_{\bar{p}} \leqslant 1$. For the spherically symmetric system we have:

$$\left.\begin{array}{ll}
\check{\psi}_{\lambda,+}(x,+) = u_\lambda\, \varphi_\lambda(x), & \check{\psi}_{\lambda,+}(x,-) = (-1)^{l+j+m}\, v_\lambda\, \varphi_{\bar{\lambda}}^*(x), \\
\check{\psi}_{\lambda,-}(x,-) = u_\lambda\, \varphi_\lambda^*(x), & \check{\psi}_{\lambda,-}(x,+) = (-1)^{l+j+m}\, v_\lambda\, \varphi_{\bar{\lambda}}(x).
\end{array}\right\} \tag{2.77}$$

Up to now we did not restrict our analysis to the systems where the number of particles is conserved exactly. The reason is that our aim was modification of the existing

general GF formalism for the arbitrary Fermi systems with pairing. However, application of the formalism to the atomic nuclei, we are interested in, implies that the particle-number conservation law is fulfilled. Thus, in what follows we assume that the total Hamiltonian $H$ defined by Eqs. (2.4)–(2.7) does not contain the external anomalous pair potentials $\Delta^0$ and that Eqs. (2.8) hold. In that case the ground-state wave function $|0\rangle$ which enters the definition of the GFs (2.12) is an eigenfunction of the particle-number operator. This means that the exact GFs do not contain anomalous components. In particular, the exact single-particle GF satisfies condition [cf. Eq. (2.45)]:

$$(\chi - \chi')\, G(z, z') = 0\,. \tag{2.78}$$

However, this condition is not fulfilled for the GF $\tilde{G}$ which is the solution of Eq. (2.61) with the operator $\tilde{\Sigma}$ including the pair potentials independently of the Hamiltonian $H$ (e.g., within the HFB approximation). Justification of using such GF $\tilde{G}$ is as follows. It enables one to take into account pairing correlations effectively and should be considered only as an approximation to the exact GF. The latter is found from Eq. (2.60) in which the mass operator $\Sigma^e$ has to contain all necessary corrections to $\tilde{\Sigma}$, such that the solution of Eq. (2.60) satisfies Eq. (2.78). Of course this scheme should be considered only as a philosophy of the approach, i.e. as an ideal program which is difficult to implement completely in practice.

Let us now define the energy representation of the Green functions and of the related quantities entering above equations. Making use of the basis $\{\psi_1(y)\}$, let us introduce the following Fourier transformations:

$$G_{12}(\varepsilon) = \int dz_1\, dz_2\, \psi_1^*(y_1)\, \psi_2(y_2)\, \delta(t_2)\, \exp\big(i\varepsilon\,(t_1 - t_2)\big)\, G(z_1, z_2)\,, \tag{2.79}$$

$$\begin{aligned}
R_{12,34}(\omega) =\ & -i \int dz_1\, dz_2\, dz_3\, dz_4\, \psi_1(y_1)\, \psi_2^*(y_2)\, \psi_3^*(y_3)\, \psi_4(y_4) \\
& \times\, \delta(t_1 - t_2 - 0)\, \delta(t_4 - t_3 - 0)\, \delta(t_4)\, \exp\big(i\omega\,(t_3 - t_1)\big) \\
& \times\, R(z_1, z_2;\, z_3, z_4)\,,
\end{aligned} \tag{2.80}$$

$$\begin{aligned}
\mathcal{U}_{12,34}(\omega,\, \varepsilon,\, \varepsilon') =\ & \int dz_1\, dz_2\, dz_3\, dz_4\, \psi_1(y_1)\, \psi_2^*(y_2)\, \psi_3^*(y_3)\, \psi_4(y_4) \\
& \times\, \delta(t_4)\, \exp\big(i\omega\,(t_3 - t_1) + i\varepsilon\,(t_2 - t_1) + i\varepsilon'\,(t_3 - t_4)\big) \\
& \times\, \mathcal{U}(z_1, z_2;\, z_3, z_4)\,.
\end{aligned} \tag{2.81}$$

The quantities $\tilde{G}_{12}(\varepsilon)$, $\Sigma_{12}(\varepsilon)$, $\Sigma_{12}^e(\varepsilon)$, $L_{12,34}(\omega)$, $\mathcal{U}_{12,34}^e(\omega,\, \varepsilon,\, \varepsilon')$, and others are defined in analogy to these formulas. In accordance with Eqs. (2.56), (2.72), and (2.80) we have:

$$L_{12,34}(\omega) = R_{12,34}(\omega) - R_{\bar{2}\bar{1},34}(\omega)\,. \tag{2.82}$$

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Notice that spectral expansion for the response function $L(\omega)$ has the form:

$$L_{12,34}(\omega) = -\sum_{\eta=\pm1}\sum_{n\neq0} \frac{\eta\,\rho_{12}^{n(\eta)}\,\rho_{34}^{n(\eta)*}}{\omega - \eta\,(\omega_n - i\cdot0)}, \tag{2.83}$$

which is similar to the analogous formula for the Fermi systems without pairing correlations (see, e.g., Ref [19]). The difference consists in the definition of the transition amplitudes. In Eq. (2.83) we have:

$$\rho_{12}^{n(\eta)} = \delta_{\eta,+1}\,\langle\,n\,|\,b_1^\dagger\,b_2\,|\,0\,\rangle + \delta_{\eta,-1}\,\langle\,0\,|\,b_1^\dagger\,b_2\,|\,n\,\rangle, \qquad \omega_n = E_n - E_0, \tag{2.84}$$

where $|\,n\,\rangle$, $|\,0\,\rangle$, $E_n$, and $E_0$ are the eigenfunctions and the eigenvalues of the Hamiltonian $H$. Taking into account the definitions (2.1) and (2.74), one can see that even if the ground-state wave function $|\,0\,\rangle$ is an eigenfunction of the particle-number operator, the amplitudes $\rho_{12}^{n(\eta)}$ take nonzero values not only for the transitions between the states with the same number of particles, but also for the transitions between the ground state of the $N$-particle system $|\,0\,\rangle$ and the states $|\,n\,\rangle$ of the systems consisting of $N\pm2$ particles. Thus, the spectral expansion (2.83) contains information about excitations of the $N$-particle system both in the ph, and in the pp channels.

# 3 QUASIPARTICLE TIME BLOCKING APPROXIMATION

## 3.1 General framework

Let us now turn to the question of determining the physical observables and related quantities in this approach, namely, excitation energies $\omega_n$ and transition amplitudes $\rho_{12}^{n(\eta)}$. It follows from the Eqs. (2.82) and (2.83) that in order to find these characteristics we need to know the response function $R(\omega)$, i.e. to solve the system of equations (2.62) and (2.64). The basic difficulty in solving this task is that the Eq. (2.64) contains energy-dependent interaction $\mathcal{U}^e$ and mass operator $\Sigma^e$. Notice, however, that these energy-dependent quantities arise only in the case when the dynamic contributions of complex configurations are taken into account explicitly. In the energy representation Eq. (2.64) is an integral equation for the function $R^e(\omega,\,\varepsilon,\,\varepsilon')$ [defined in analogy to Eq. (2.81)] over the energy variable $\varepsilon$, which cannot be, strictly speaking, reduced to the closed equation for $R^e(\omega)$ due to the energy dependence of $\mathcal{U}^e$ and $\Sigma^e$. Fortunately, there are methods that allow us to avoid complicated problem of exact solution of this equation, making use of certain approximations. One of such methods will be considered here.

We begin by noting that if we use the eigenfunctions of the operator $\mathcal{H}$, i.e. the set $\{\psi_1(y)\}$, as the basis functions, the single-particle GF $\tilde{G}$ is diagonal:

$$\tilde{G}_{12}(\varepsilon) = \frac{\delta_{12}}{\varepsilon - E_1 + i\,\eta_1\cdot 0}\,, \tag{3.1}$$

as follows from Eqs. (2.24), (2.61), (2.69), and (2.70). In the time representation we have:

$$\tilde{G}_{12}(t_1, t_2) = -i\,\eta_1\delta_{12}\,\theta(\eta_1 t_{12})\,\exp(-iE_1 t_{12})\,, \tag{3.2}$$

where $t_{12} = t_1 - t_2$, $\theta(\tau)$ is the step function. These expressions are formally identical with analogous formulas for the normal GFs, except that they are written in the extended basis representation. It enables one to apply, practically without changes, the method of chronological decoupling of diagrams (MCDD) to the solution of Eqs. (2.62) and (2.64) which contain the GFs with pairing. The MCDD was developed in Ref. [10] for the solution of the ph-channel BSE in the normal Fermi system including dynamic effects both in the interaction and in the mass operator. The idea of the method is similar to that used in the other methods developed earlier for the solution of the analogous problems in Refs. [20] (ph-channel BSE) and [21, 22, 23] (pp-channel BSE). However, the MCDD differs from the aforementioned methods in some details, in particular concerning the treatment of the GSC2. Almost all the resulting equations obtained by means of this straightforward extension of the MCDD are found to be formally identical with equations for the normal Fermi system in the same sense as Eqs. (3.1) and (3.2). Because derivation of these equations in the latter case was described in detail in Refs. [3, 10], we will draw only the main formulas and the final results.

First of all, the function $\tilde{R}^0 = -\tilde{G}\tilde{G}$ entering Eq. (2.64) is divided into two parts: $\tilde{R}^0 = \tilde{R}^{0(a)} + \tilde{R}^{0(b)}$ where

$$\tilde{R}^{0(a)}_{12,34}(t_1, t_2\,;\; t_3, t_4) = -\delta_{\eta_1, -\eta_2}\,\theta(\eta_1 t_{41})\,\theta(\eta_1 t_{32})\,\tilde{G}_{31}(t_3, t_1)\,\tilde{G}_{24}(t_2, t_4)\,, \tag{3.3}$$

$\tilde{R}^{0(b)}$ is the remainder term, which is absorbed in part in the renormalization procedure. As compared with the initial function $\tilde{R}^0$ defined by Eq. (2.65), the term $\tilde{R}^{0(a)}$ contains two additional time-dependent step functions and the factor $\delta_{\eta_1, -\eta_2}$ which play twofold role. On the one hand, they allow to obtain closed set of the algebraic equations in the energy representation for the main component of the function $R^e(\omega)$ [see Eqs. (3.5), (3.17), (3.18) below] which is much more simple for the solution as compared with the initial Eq. (2.64). On the other hand, owing to these additional $\theta$ and $\delta$ functions, an approximate solution

of the Eq. (2.64) obtained by this way contains the contributions of the 2q and 2q⊗phonon configurations, while more complicated intermediate states (e.g., 2q⊗2phonon, 2q⊗3phonon, and so on) are blocked, in part, in the time representation. So in the following this scheme will be referred to as the quasiparticle time blocking approximation (QTBA).

Further, renormalization procedure is applied to Eq. (2.62) for the response function $R$, which leads to the following equation for the effective response function $R^{\text{eff}}$ in the energy representation:

$$R^{\text{eff}}_{12,34}(\omega) = A_{12,34}(\omega) - \sum_{5678} A_{12,56}(\omega)\, \mathcal{F}_{56,78}\, R^{\text{eff}}_{78,34}(\omega)\,, \tag{3.4}$$

where $A(\omega)$ is a joint (ph and pp) correlated propagator, $\mathcal{F}$ is an amplitude of the effective interaction. The propagator $A(\omega)$ is the main term of the formal decomposition

$$R^{e}_{12,34}(\omega) = A_{12,34}(\omega) + B_{12,34}\,. \tag{3.5}$$

It contains: (i) the sum of an infinite number of terms to all orders in $\tilde{R}^{0(a)}$, and (ii) the terms linear and quadratic in $\tilde{R}^{0(b)}$ which are related to the GSC2. The term $B$ in Eq. (3.5) is an auxiliary quantity which is supposed to be energy-independent. In addition, it is supposed that $B$ contains all the contributions which are not included explicitly in the propagator $A(\omega)$. The effective interaction $\mathcal{F}$ and the effective charge operator $e$ are defined by the formulas:

$$\mathcal{F}_{12,34} = \sum_{56} e_{12,56}\, \tilde{\mathcal{U}}_{56,34}\,, \qquad e_{12,34} = \delta_{13}\,\delta_{24} - \sum_{56} \mathcal{F}_{12,56}\, B_{56,34} = (e^{\dagger}_{34,12})^{*}\,. \tag{3.6}$$

In terms of these quantities the exact response function $R$ is related to the effective response function by the ansatz:

$$R_{12,34}(\omega) = \sum_{5678} e^{\dagger}_{12,56}\, R^{\text{eff}}_{56,78}(\omega)\, e_{78,34} + \sum_{56} B_{12,56}\, e_{56,34}\,. \tag{3.7}$$

One of the basic quantities, which determines the physical observables in this approach, is the nuclear polarizability $\Pi(\omega)$. More precisely, it determines distribution of the transition strength caused by an external field $V^{0}(x,x')$. The function $\Pi(\omega)$ is defined as

$$\Pi(\omega) = -\frac{1}{2}\sum_{1234}(e\,V^{0})^{*}_{21}\, R^{\text{eff}}_{12,34}(\omega)\,(e\,V^{0})_{43}\,, \qquad (e\,V^{0})_{12} = \sum_{34} e_{21,43}\, V^{0}_{34}\,, \tag{3.8}$$

where

$$V^{0}_{12} = \int dy\, dy'\, \psi^{*}_{1}(y)\, \psi_{2}(y')\, \delta_{\chi,\chi'}\left[\delta_{\chi,+1}\, V^{0}(x,x') - \delta_{\chi,-1}\, V^{0}(x',x)\right] = -V^{0}_{2\bar{1}}\,. \tag{3.9}$$

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In particular, the strength function $S(E)$ which is frequently used for the description of nuclear excitations is expressed in terms of the polarizability as

$$S(E) = \frac{1}{2\pi} \operatorname{Im} \sum_{1234} V_{21}^{0*} R_{12,34}(E + i\,\Delta) V_{43}^0 = -\frac{1}{\pi} \operatorname{Im} \Pi(E + i\,\Delta), \qquad (3.10)$$

where $\Delta$ is a smearing parameter. The formulas (3.4)–(3.8), (3.10) are completely analogous to the ones for the normal Fermi system (see Ref. [3]), except for the factors $\frac{1}{2}$ in Eqs. (3.8) and (3.10) which arise due to definition (3.9) of the operator $V^0$ in the extended space taken in the antisymmetric form.

## 3.2 Correlated propagator within the QTBA

Eq. (3.4) for the response function is still quite general. To formulate a model we have to define the correlated propagator $A(\omega)$. In particular, if we neglect the dynamic contributions of complex configurations, i.e. if we put $\mathcal{U}^e = 0$ and $\Sigma^e = 0$, we come to the QRPA. In this case we have: $A(\omega) = \tilde{A}(\omega)$ where $\tilde{A}(\omega)$ is uncorrelated QRPA propagator:

$$\tilde{A}_{12,34}(\omega) = -\frac{\eta_1 \, \delta_{\eta_1, -\eta_2} \, \delta_{13} \, \delta_{24}}{\omega - E_{12}}, \qquad E_{12} = E_1 - E_2. \qquad (3.11)$$

To go beyond the QRPA we have to find reasonable approximations for the quantities $\mathcal{U}^e$ and $\Sigma^e$. In the present work we shall use a QRPA-based version of the quasiparticle-phonon coupling (QPC) model (see Ref. [24]). This model is discussed and used in a variety of papers: see, e.g., Refs. [2, 3, 7, 12, 19]. Within the QPC model one can restrict oneself to so-called $g^2$ approximation where $g$ is an amplitude of the quasiparticle-phonon interaction (see Ref. [3] for more details). Under some simplifying assumptions, this approximation can be obtained in the GF method (see Refs. [25, 26, 27]). Within the QPC model and $g^2$ approximation we have the following formulas for the quantities $\mathcal{U}^e$ and $\Sigma^e$:

$$\mathcal{U}_{12,34}^e(\omega, \varepsilon, \varepsilon') = \sum_{\eta, m} \frac{\eta \, g_{31}^{m(\eta)*} \, g_{42}^{m(\eta)}}{\varepsilon - \varepsilon' + \eta \, (\omega_m - i \cdot 0)}, \qquad (3.12)$$

$$\Sigma_{12}^e(\varepsilon) = \sum_{3, \eta, m} \frac{\delta_{\eta, \eta_3} \, g_{13}^{m(\eta)*} \, g_{23}^{m(\eta)}}{\varepsilon - E_3 - \eta \, (\omega_m - i \cdot 0)}, \qquad (3.13)$$

where $\eta = \pm 1$. Hereafter it is assumed that the quasiparticle-phonon amplitudes $g_{12}^{m(\eta)}$ are related to the transition amplitudes $\rho_{12}^{m(\eta)}$ by means of QRPA equations:

$$g_{12}^{m(\eta)} = \sum_{34} \tilde{\mathcal{F}}_{12,34} \, \rho_{34}^{m(\eta)}, \qquad \rho_{12}^{m(\eta)} = \frac{\eta_1 \, \delta_{\eta_1, -\eta_2}}{\eta \, \omega_m - E_{12}} \, g_{12}^{m(\eta)}, \qquad (3.14)$$

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where $\tilde{\mathcal{F}}$ is an amplitude of the effective interaction which generally differs from the amplitude $\mathcal{F}$ entering Eq. (3.4). Notice that the QRPA equations acquire very simple form in the representation of single-quasiparticle $\psi$-functions. The Eqs. (3.14) have to be supplemented by the normalization condition

$$\frac{1}{2} \sum_{12} \eta\, \eta_1 \left| \rho_{12}^{m(\eta)} \right|^2 = 1 \qquad (3.15)$$

and by the condition of the antisymmetry

$$\rho_{12}^{m(\eta)} = -\rho_{\bar{2}\bar{1}}^{m(\eta)} \qquad (3.16)$$

which is obviously fulfilled for the exact transition amplitudes defined by Eq. (2.84). Notice, however, that in contrast to the case considered in Refs. [3, 10], the quasiparticle-phonon amplitudes $g_{12}^{m(\eta)}$ in Eqs. (3.12) and (3.13) determine the coupling with excitations both in the ph, and in the pp channels [see Eqs. (3.14) and comments after Eq. (2.84)].

The QTBA in combination with Eqs. (3.12) and (3.13) leads to the following ansatz for the correlated propagator:

$$A_{12,34}(\omega) = \sum_{5678} \left[ \delta_{15}\, \delta_{26} + Q_{12,56}^{(+-)}(\omega) \right] A_{56,78}^{(--)}(\omega) \left[ \delta_{73}\, \delta_{84} + Q_{78,34}^{(-+)}(\omega) \right] + P_{12,34}^{(++)}(\omega), \qquad (3.17)$$

where the upper indices denote products of the first and the second pairs of lower $\eta$-indices. In particular, for the component $A_{12,34}^{(--)}(\omega)$ of the propagator we have: $\eta_1\eta_2 = \eta_3\eta_4 = -1$. This component is determined by the equation

$$A_{12,34}^{(--)}(\omega) = \tilde{A}_{12,34}(\omega) - \sum_{5678} \tilde{A}_{12,56}(\omega)\, \Phi_{56,78}(\omega)\, A_{78,34}^{(--)}(\omega), \qquad (3.18)$$

where $\tilde{A}(\omega)$ is the QRPA propagator defined by Eq. (3.11). For the remaining quantities in the Eqs. (3.17) and (3.18) we obtain

$$Q_{12,34}^{(+-)}(\omega) = Q_{12,34}^{(+-)\,\mathrm{res}}(\omega) + \delta_{\eta_1,\eta_2}\, \delta_{\eta_3,-\eta_4} \left( \frac{\Sigma_{31}^{\mathrm{GSC}}}{E_{31}} \delta_{24} - \delta_{31} \frac{\Sigma_{24}^{\mathrm{GSC}}}{E_{24}} \right), \qquad (3.19)$$

$$Q_{12,34}^{(-+)}(\omega) = Q_{12,34}^{(-+)\,\mathrm{res}}(\omega) - \delta_{\eta_1,-\eta_2}\, \delta_{\eta_3,\eta_4} \left( \frac{\Sigma_{31}^{\mathrm{GSC}}}{E_{31}} \delta_{24} - \delta_{31} \frac{\Sigma_{24}^{\mathrm{GSC}}}{E_{24}} \right), \qquad (3.20)$$

$$\Phi_{12,34}(\omega) = \Phi_{12,34}^{\mathrm{res}}(\omega) + \bar{\Phi}_{12,34}^{\mathrm{GSC}} + \Phi_{12,34}^{\mathrm{GSC}\ s.e.}(\omega). \qquad (3.21)$$

In these formulas, a superscript "res" denotes the resonant parts of the amplitudes, the quantities $\bar{\Phi}^{\mathrm{GSC}}$ and $\Phi^{\mathrm{GSC}\ s.e.}(\omega)$ represent contributions of the GSC. They consist of the

static part arising from the induced interaction ($\bar{\Phi}^{\text{GSC}}$) and of the part arising from the self-energy insertions ($\Phi^{\text{GSC}\,s.e.}$):

$$\bar{\Phi}^{\text{GSC}}_{12,34} = -\delta_{\eta_1,-\eta_2}\,\delta_{\eta_3,-\eta_4}\sum_{\eta,m}\left(\delta_{\eta,\eta_3}\,\rho^{m(\eta)}_{13}\,g^{m(\eta)*}_{24} + \delta_{\eta,\eta_4}\,g^{m(\eta)}_{13}\,\rho^{m(\eta)*}_{24}\right), \quad (3.22)$$

$$\Phi^{\text{GSC}\,s.e.}_{12,34}(\omega) = \eta_1\,\delta_{\eta_1,-\eta_2}\,\delta_{\eta_3,-\eta_4}\left(\Sigma^{\text{GSC}}_{31}\left(\delta_{24}+q_{24}\right) - \left(\delta_{31}+q_{31}\right)\Sigma^{\text{GSC}}_{24}\right.$$
$$\left. - \left(q_{31}\,\delta_{24}+\delta_{31}\,q_{24}+q_{31}\,q_{24}\right)\left[\omega-\tfrac{1}{2}\left(E_{12}+E_{34}\right)\right]\right), \quad (3.23)$$

where

$$\Sigma^{\text{GSC}}_{12} = \tfrac{1}{2}\left(1+\delta_{\eta_1,-\eta_2}\right)\sum_{3,\eta,m}\eta\,\delta_{\eta,\eta_3}\left(\rho^{m(\eta)*}_{13}\,g^{m(\eta)}_{23} + g^{m(\eta)*}_{13}\,\rho^{m(\eta)}_{23}\right), \quad (3.24)$$

$$q_{12} = \sum_{3,\eta,m}\delta_{\eta,\eta_3}\,\rho^{m(\eta)*}_{13}\,\rho^{m(\eta)}_{23}. \quad (3.25)$$

The component $P^{(++)}(\omega)$ of the correlated propagator and the resonant parts of the amplitudes entering Eqs. (3.19)–(3.21) are defined as

$$P^{(++)}_{12,34}(\omega) = \sum_{5678,\eta,m}\zeta^{m56(\eta)}_{12}\,\tilde{A}^{(\eta)}_{56,78}(\omega-\eta\,\omega_m)\,\zeta^{m78(\eta)*}_{34}, \quad (3.26)$$

$$Q^{(+-)\,\text{res}}_{12,34}(\omega) = \sum_{5678,\eta,m}\zeta^{m56(\eta)}_{12}\,\tilde{A}^{(\eta)}_{56,78}(\omega-\eta\,\omega_m)\,\gamma^{m78(\eta)*}_{34}, \quad (3.27)$$

$$Q^{(-+)\,\text{res}}_{12,34}(\omega) = \sum_{5678,\eta,m}\gamma^{m56(\eta)}_{12}\,\tilde{A}^{(\eta)}_{56,78}(\omega-\eta\,\omega_m)\,\zeta^{m78(\eta)*}_{34}, \quad (3.28)$$

$$\Phi^{\text{res}}_{12,34}(\omega) = -\sum_{5678,\eta,m}\gamma^{m56(\eta)}_{12}\,\tilde{A}^{(\eta)}_{56,78}(\omega-\eta\,\omega_m)\,\gamma^{m78(\eta)*}_{34}, \quad (3.29)$$

where

$$\gamma^{m56(\eta)}_{12} = \delta_{\eta,\eta_5}\,\delta_{\eta_1,-\eta_2}\,\delta_{\eta_5,-\eta_6}\left(\delta_{15}\,g^{m(\eta)}_{62} - g^{m(\eta)}_{15}\,\delta_{62}\right), \quad (3.30)$$

$$\zeta^{m56(\eta)}_{12} = \delta_{\eta,\eta_5}\,\delta_{\eta_1,\eta_2}\,\delta_{\eta_5,-\eta_6}\left(\delta_{15}\,\rho^{m(\eta)}_{62} - \rho^{m(\eta)}_{15}\,\delta_{62}\right), \quad (3.31)$$

$\tilde{A}^{(+)}(\omega)$ and $\tilde{A}^{(-)}(\omega)$ are the positive and the negative frequency parts of the QRPA propagator defined by Eq. (3.11), i.e.: $\tilde{A}(\omega) = \tilde{A}^{(+)}(\omega) + \tilde{A}^{(-)}(\omega)$,

$$\tilde{A}^{(\eta)}_{12,34}(\omega) = -\frac{\eta\,\delta_{\eta,\eta_1}\,\delta_{\eta_1,-\eta_2}\,\delta_{13}\,\delta_{24}}{\omega-\eta\,\eta_1\,E_{12}}. \quad (3.32)$$

Correlated propagator $A(\omega)$ defined by Eq. (3.17) and subsequent equations includes contributions of three types: (i) pure 2q configurations associated with uncorrelated QRPA propagator $\tilde{A}(\omega)$, (ii) 2q⊗phonon configurations introduced by the quantities $\mathcal{U}^e$ and $\Sigma^e$, and (iii) uncontrollable more complicated configurations arising due to the GSC effects and their combinations with the above-mentioned configurations.

## 3.3 Sum rule analysis and a refinement of the model

The formulas of the previous subsection completely determine the correlated propagator of the model within the $g^2$ approximation. By construction, this propagator contains all the $g^2$ contributions, including those from GSC. However, exact fulfillment of the sum rules in this approach is not guaranteed. Let us consider this question in more detail. Usually, the sum rule is meant in the sense of relation between the moment $m_k$ of the strength function $S(E)$ and the ground state expectation value of certain operator (see, e.g., Ref. [28]). The moment $m_k$ is defined as

$$m_k = \frac{1}{2} \int_{-\infty}^{\infty} S(E)\, E^k dE \tag{3.33}$$

at $\Delta \to +0$ in Eq. (3.10). Introducing asymptotic expansion of the exact response function:

$$R_{12,34}(\omega) \sim -\sum_{k=0}^{\infty} M_{12,34}^{(k)}\, \omega^{-k-1} \tag{3.34}$$

and using Eqs. (2.82), (2.83), (3.9), (3.10), one can show that the moments $m_k$ are expressed through the coefficient functions $M_{12,34}^{(k)}$ by the formula:

$$m_k = \frac{1}{4} \sum_{1234} V_{21}^{0*}\, M_{12,34}^{(k)}\, V_{43}^0\,. \tag{3.35}$$

In particular, making use of the BSE (2.53) in the energy representation one can obtain:

$$M_{12,34}^{(0)} = \delta_{31}\, \rho_{24} - \rho_{31}\, \delta_{24}\,, \tag{3.36}$$

where $\rho_{12} = \langle 0\,|\, b_2^\dagger b_1\,|\, 0 \rangle$ is the extended density matrix (EDM). Substituting Eq. (3.36) into the Eq. (3.35) we get the so-called non-energy-weighted sum rule (NEWSR):

$$m_0 = \tfrac{1}{4} \mathrm{Tr}\big( \rho\, [V^0, V^{0\dagger}] \big)\,. \tag{3.37}$$

Notice that the factor $\frac{1}{4}$ in this formula (instead of the usual factor $\frac{1}{2}$) arises from the definition (3.9) of the external field operator in the extended space taken in the antisymmetric form.

Thus, in order to ensure exact fulfillment of the NEWSR the coefficient function $M_{12,34}^{(0)}$ of the model must have the form (3.36) with properly normalized EDM $\rho$. Using the formulas of the previous subsection and the definition of the quantity $M_{12,34}^{(0)}$ through the expansion (3.34) one can show that Eq. (3.36) is not fulfilled in the model considered and that the NEWSR is fulfilled only up to within the terms of order $g^4$.

It is not difficult, however, to remedy this drawback within the above-described scheme based on the $g^2$ approximation. First of all, let us include energy-independent operator $\Sigma^{\mathrm{GSC}}$ defined by Eq. (3.24) into the mean-field part $\tilde{\Sigma}$ of the total mass operator $\Sigma(\varepsilon)$. It can be done because the only constraint on the operator $\tilde{\Sigma}$ was the condition of its energy independence. This redefinition of $\tilde{\Sigma}$ means that we have to use in all the equations the operator $\bar{\Sigma}^e(\varepsilon) = \Sigma^e(\varepsilon) - \Sigma^{\mathrm{GSC}}$ instead of $\Sigma^e(\varepsilon)$. The replacement of $\Sigma^e(\varepsilon)$ by $\bar{\Sigma}^e(\varepsilon)$ leads to disappearance of all the terms containing $\Sigma^{\mathrm{GSC}}$ in Eqs. (3.19), (3.20), and (3.23). The remaining part of the amplitude $\Phi^{\mathrm{GSC}\, s.e.}(\omega)$ in Eq. (3.21) can be taken into account through the renormalization of the QRPA propagator $\tilde{A}(\omega)$ within the $g^2$ approximation.

To this aim let us introduce matrix $\tilde{Z}_{12,34}$ defined by the following equations:

$$\sum_{56} \tilde{Z}_{12,56}\, \tilde{Z}_{56,34} = \delta_{31}\, \delta_{24} - q_{31}\, \delta_{24} - \delta_{31}\, q_{24}\,, \qquad \tilde{Z}_{12,34} = \tilde{Z}^*_{34,12}\,, \tag{3.38}$$

where the matrix $q_{12}$ is defined by Eq. (3.25). In addition it will be supposed that the matrix $\tilde{Z}_{12,34}$ is positive-definite that can be always fulfilled if all the eigenvalues $q_i$ of the matrix $q_{12}$ satisfy condition: $q_i < \frac{1}{2}$ [notice that $q_i \geqslant 0$, as follows from Eq. (3.25)]. Because according to Eqs. (3.14) and (3.25) we have: $q_{12} = O(g^2)$, the pointed condition is consistent with the previous model assumptions. Thus, from Eq. (3.38) it follows that

$$\tilde{Z}_{12,34} = \delta_{31}\, \delta_{24} - \tfrac{1}{2}\left( q_{31}\, \delta_{24} + \delta_{31}\, q_{24} \right) + O(g^4)\,. \tag{3.39}$$

Further, using Eqs. (3.11), (3.23) (without terms containing operator $\Sigma^{\mathrm{GSC}}$), and (3.39), we obtain:

$$\tilde{A}(\omega) - \tilde{A}(\omega)\, \Phi^{\mathrm{GSC}\, s.e.}(\omega)\, \tilde{A}(\omega) = \tilde{Z}\, \tilde{A}(\omega)\, \tilde{Z} + O(g^4)\,. \tag{3.40}$$

It enables one to redefine correlated propagator replacing Eq. (3.17) by the following ansatz:

$$A_{12,34}(\omega) = \sum_{5678} Z^L_{12,56}(\omega)\, A^{(--)}_{56,78}(\omega)\, Z^R_{78,34}(\omega) + P^{(++)}_{12,34}(\omega)\,, \tag{3.41}$$

where

$$Z^L_{12,34}(\omega) = \sum_{56} \left[ \delta_{15}\, \delta_{26} + Q^{(+-)\,\mathrm{res}}_{12,56}(\omega) \right] \tilde{Z}_{56,34}\,, \tag{3.42}$$

$$Z^R_{12,34}(\omega) = \sum_{56} \tilde{Z}_{12,56} \left[ \delta_{53}\, \delta_{64} + Q^{(-+)\,\mathrm{res}}_{56,34}(\omega) \right]. \tag{3.43}$$

In Eq. (3.41) the propagator $A^{(--)}(\omega)$ is determined by Eq. (3.18) in which the amplitude $\Phi(\omega)$ is now defined as

$$\Phi_{12,34}(\omega) = \sum_{5678} \tilde{Z}_{12,56} \left[ \Phi_{56,78}^{\mathrm{res}}(\omega) + \bar{\Phi}_{56,78}^{\mathrm{GSC}} \right] \tilde{Z}_{78,34}, \tag{3.44}$$

instead of the Eq. (3.21).

It is easy to see that the propagator $A(\omega)$ defined by Eqs. (3.18), (3.38), (3.41)–(3.44) coincides up to within the terms of order $g^4$ with the propagator defined in the previous subsection [see Eq. (3.17) and subsequent equations]. On the other hand, assuming that the effective charge in Eq. (3.7) is equal to unit operator and making use of the expansion (3.34), one can find that this new redefined propagator leads to the following result for the coefficient function $M_{12,34}^{(0)}$:

$$M_{12,34}^{(0)} = \eta_1 \, \delta_{\eta_1,-\eta_2} \left( \delta_{31} \, \delta_{24} - q_{31} \, \delta_{24} - \delta_{31} \, q_{24} \right) + \eta_1 \, \delta_{\eta_1,\eta_2} \left( \delta_{31} \, q_{24} - q_{31} \, \delta_{24} \right). \tag{3.45}$$

Here the first term containing $\delta_{\eta_1,-\eta_2}$ follows from Eq. (3.38). The second term arises from the contribution of the component $P^{(++)}(\omega)$ in Eq. (3.41). From the Eq. (3.45) we obtain that in the modified version of the model the coefficient function $M_{12,34}^{(0)}$ has the form (3.36) with correlated EDM $\rho$ defined as

$$\rho_{12} = \tilde{\rho}_{12} + \eta_1 \, q_{12}, \tag{3.46}$$

where $\tilde{\rho}_{12} = \delta_{\eta_1,-1} \delta_{12}$ is the EDM of the HFB theory in the representation of $\psi$-functions. Therefore we conclude that if the EDM (3.46) is normalized by the usual condition

$$\int dy' \, \delta_{\chi',+1} \, \delta_{\tau',\tau} \sum_{12} \psi_1(y') \, \psi_2^*(y') \, \rho_{12} = N_\tau, \tag{3.47}$$

the NEWSR is fulfilled exactly within the QTBA. It is worthwhile mentioning that the EDM (3.46) arising in the QTBA coincides with the correlated EDM $\rho^c$ which can be obtained from the Dyson equation (2.60) and the Eqs. (3.1), (3.13), (3.24) within $g^2$ approximation:

$$\rho_{12} = \rho_{12}^c \equiv \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi i} \, e^{i\varepsilon\tau} \left[ \tilde{G}(\varepsilon) + \tilde{G}(\varepsilon) \left( \Sigma^e(\varepsilon) - \Sigma^{\mathrm{GSC}} \right) \tilde{G}(\varepsilon) \right]_{12}, \qquad \tau \to +0. \tag{3.48}$$

Finally, note that the GSC effects included by means of renormalization of the QRPA propagator within the QTBA with the help of the matrix $\tilde{Z}$ are the same as the effects included in the renormalized QRPA (RQRPA, see, e.g., Ref. [6]). It is known (see Ref. [29]) that within the standard RQRPA, the Ikeda sum rule (being a particular case of the NEWSR)

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is violated. The above analysis allows us to understand the reason of this violation. It follows from Eq. (3.45) that to satisfy the NEWSR within the QTBA it is necessary to take into account contribution of the term $P^{(++)}(\omega)$. This term represents dynamic contributions of the GSC into the QTBA propagator (3.41), which cannot be reduced to the renormalization of the QRPA propagator $\tilde{A}(\omega)$ and which are absent in the RQRPA.

## 3.4 Antisymmetrization of the equations and inclusion of the two-phonon configurations

As can be seen from the spectral expansion (2.83), the physical observables of the theory are completely determined by the antisymmetric response function $L(\omega)$. Within the QTBA this exact function is approximated by the effective antisymmetric response function $L^{\mathrm{eff}}(\omega)$ defined as [cf. Eq. (2.82)]

$$L^{\mathrm{eff}}_{12,34}(\omega) = R^{\mathrm{eff}}_{12,34}(\omega) - R^{\mathrm{eff}}_{\bar{2}\bar{1},34}(\omega), \qquad (3.49)$$

where $R^{\mathrm{eff}}(\omega)$ is the solution of Eq. (3.4). It is easy to prove the following. First, the polarizability $\Pi(\omega)$ [see Eqs. (3.8)] is actually determined by the function $L^{\mathrm{eff}}(\omega)$, while symmetric part of the function $R^{\mathrm{eff}}(\omega)$ does not contribute to the Eqs. (3.8). In other words, $\Pi(\omega)$ is invariant under the transformation: $R^{\mathrm{eff}}(\omega) \rightarrow \frac{1}{2} L^{\mathrm{eff}}(\omega)$. Second, the function $L^{\mathrm{eff}}(\omega)$ is the doubled solution of the antisymmetrized QTBA equation obtained from the Eq. (3.4) with the help of antisymmetrization of the correlated propagator $A(\omega)$. This antisymmetrization can be implemented by means of the following transformations in Eqs. (3.18), (3.26)–(3.29): $\tilde{A}(\omega) \rightarrow \frac{1}{2} \tilde{L}^0(\omega)$, $\tilde{A}^{(\eta)}(\omega) \rightarrow \frac{1}{2} \tilde{L}^{0(\eta)}(\omega)$, where $\tilde{L}^0(\omega)$ is the antisymmetric (uncorrelated) QRPA propagator, $\tilde{L}^{0(\eta)}(\omega)$ represents its positive and negative frequency parts [cf. Eqs. (3.11) and (3.32)]:

$$\tilde{L}^0_{12,34}(\omega) = \sum_{\eta=\pm 1} \tilde{L}^{0(\eta)}_{12,34}(\omega), \qquad \tilde{L}^{0(\eta)}_{12,34}(\omega) = \left(\delta_{\bar{2}3}\,\delta_{\bar{1}4} - \delta_{13}\,\delta_{24}\right) \frac{\eta\,\delta_{\eta,\eta_1}\,\delta_{\eta_1,-\eta_2}}{\omega - \eta\,\eta_1\,E_{12}}. \qquad (3.50)$$

We did not use the antisymmetric form of the QTBA equations from the very beginning to simplify their derivation and analysis. However, the antisymmetrization facilitates numerical solution due to reduction of the dimensions of matrices entering these equations.

The model described above allows for the following straightforward extension related to the definition of the resonant parts of the amplitudes entering Eqs. (3.18), (3.41)–(3.44) for the correlated propagator of the QTBA. Contributions from these resonant parts [defined by

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Eqs.(3.26)–(3.29)] to the response function describe simultaneous propagation of the phonon and of the uncorrelated quasiparticle pair. Natural generalization of this model is inclusion of the correlations in the quasiparticle pair entering 2q⊗phonon configuration, i.e. replacement of the uncorrelated pair by the phonon. For the ph-channel BSE in the normal Fermi system similar generalization, corresponding to the replacement of the 1p1h⊗phonon configurations by the two-phonon intermediate states, was discussed in Ref. [30]. For the pp-channel BSE analogous procedure has been implemented in Ref. [22].

Within the QTBA, two-quasiparticle correlations in the 2q⊗phonon intermediate states (i.e., two-phonon configurations) can be included in the following way. Correlated counterpart of the above-defined quantity $\tilde{L}^{0(\eta)}(\omega)$ is $\tilde{L}^{(\eta)}(\omega)$ representing the positive and the negative frequency parts of the antisymmetric QRPA response function $\tilde{L}(\omega)$:

$$\tilde{L}_{12,34}(\omega) = \sum_{\eta=\pm 1} \tilde{L}^{(\eta)}_{12,34}(\omega) \,, \qquad \tilde{L}^{(\eta)}_{12,34}(\omega) = -\sum_n \frac{\eta \, \rho_{12}^{n(\eta)} \, \rho_{34}^{n(\eta)*}}{\omega - \eta \, \omega_n} \,. \tag{3.51}$$

In the Eq. (3.51) it is supposed that the QRPA energies $\omega_n$ and transition amplitudes $\rho_{12}^{n(\eta)}$ satisfy Eqs. (3.14)–(3.16). The above considerations imply that transition to the two-phonon configurations within the QTBA can be accomplished by means of the replacement $\tilde{A}^{(\eta)}(\omega) \rightarrow \frac{1}{2} \tilde{L}^{(\eta)}(\omega)$ in Eqs. (3.26)–(3.29), which leads to the following result:

$$P^{(++)}_{12,34}(\omega) = -\frac{1}{2} \sum_{\eta,m,n} \frac{\eta \, \zeta_{12}^{mn(\eta)} \, \zeta_{34}^{mn(\eta)*}}{\omega - \eta \, \omega_{mn}} \,, \tag{3.52}$$

$$Q^{(+-)\,\mathrm{res}}_{12,34}(\omega) = -\frac{1}{2} \sum_{\eta,m,n} \frac{\eta \, \zeta_{12}^{mn(\eta)} \, \gamma_{34}^{mn(\eta)*}}{\omega - \eta \, \omega_{mn}} \,, \tag{3.53}$$

$$Q^{(-+)\,\mathrm{res}}_{12,34}(\omega) = -\frac{1}{2} \sum_{\eta,m,n} \frac{\eta \, \gamma_{12}^{mn(\eta)} \, \zeta_{34}^{mn(\eta)*}}{\omega - \eta \, \omega_{mn}} \,, \tag{3.54}$$

$$\Phi^{\mathrm{res}}_{12,34}(\omega) = \frac{1}{2} \sum_{\eta,m,n} \frac{\eta \, \gamma_{12}^{mn(\eta)} \, \gamma_{34}^{mn(\eta)*}}{\omega - \eta \, \omega_{mn}} \,, \tag{3.55}$$

where $\omega_{mn} = \omega_m + \omega_n$,

$$\gamma_{12}^{mn(\eta)} = \sum_{56} \gamma_{12}^{m56(\eta)} \, \rho_{56}^{n(\eta)} \,, \qquad \zeta_{12}^{mn(\eta)} = \sum_{56} \zeta_{12}^{m56(\eta)} \, \rho_{56}^{n(\eta)} \,. \tag{3.56}$$

Physical arguments in favor of using Eqs. (3.52)–(3.55) instead of (3.26)–(3.29) are clear. Notice, however, that the derivation of the Eqs. (3.52)–(3.55) has not been rigorous. It enables one only to assert that these formulas recover the original Eqs. (3.26)–(3.29) in

the limit of vanishing quasiparticle interaction. One can also show, using the completeness of the set of QRPA transition amplitudes, that the NEWSR is fulfilled exactly within two-phonon version of the QTBA. The rigorous derivation of the Eqs. (3.52)–(3.55) is based on the inclusion of the additional (third order in the quasiparticle interaction) contributions into the dynamic amplitude $\mathcal{U}^e$ defined by Eq. (3.12) and will not be considered here. It is worth noting that inclusion of the two-phonon configurations in Eqs. (3.52)–(3.55) brings the model closer to the QPM [2]. Comparing the QTBA and the QPM one can infer that treatment of the GSC within the QTBA is more consistent. A more detailed comparison of these models is beyond the scope of the present paper.

## 3.5 Self-consistent scheme

Finally, we briefly outline the scheme which enables one to eliminate spurious states within the QTBA. These states, being a common problem of the microscopic theories, are associated with existence of the non-trivial external field operators $V^0$ satisfying the condition: $[H, V^0] = [H^0, V^0]$, where $H$ and $H^0$ are the total and the single-particle Hamiltonian, correspondingly [see Eq. (2.4)]. Elimination of the spurious states within a consistent theory implies that they must have zero excitation energy. In terms of the GF method this means that the exact response function $R(\omega)$ must have a pole at $\omega = 0$ corresponding to the spurious states. In particular, it is well known that the QRPA response function $\tilde{R}(\omega)$ satisfying the equation

$$\tilde{R}(\omega) = \tilde{A}(\omega) - \tilde{A}(\omega)\, \tilde{\mathcal{F}}\, \tilde{R}(\omega) \tag{3.57}$$

has such a pole at $\omega = 0$ if the interaction amplitude $\tilde{\mathcal{F}}$ is related to the mean-field operator $\tilde{\Sigma}$ by the self-consistency condition

$$\tilde{\mathcal{F}} = \delta\, \tilde{\Sigma}\, /\, \delta\, \tilde{\rho}\,. \tag{3.58}$$

In the QTBA the situation is more complicated since the correlated propagator $A(\omega)$ in Eq. (3.4) has no simple structure of the QRPA propagator $\tilde{A}(\omega)$. To avoid this difficulty let us note that the exact solution of the Eq. (3.4) with the propagator $A(\omega)$ defined by Eqs. (3.41), (3.18) can be represented in the form:

$$R^{\text{eff}}(\omega) = \left[\, 1 - P^{(++)}(\omega)\, \mathcal{F}^P(\omega)\, \right] Z^L(\omega)\, \tilde{R}^{\text{eff}}(\omega)\, Z^R(\omega) \left[\, 1 - \mathcal{F}^P(\omega)\, P^{(++)}(\omega)\, \right]$$
$$+ P^{(++)}(\omega) - P^{(++)}(\omega)\, \mathcal{F}^P(\omega)\, P^{(++)}(\omega)\,, \tag{3.59}$$

where energy-dependent interaction amplitude $\mathcal{F}^P(\omega)$ and the renormalized response function $\tilde{R}^{\text{eff}}(\omega)$ satisfy the equations:

$$\mathcal{F}^P(\omega) \;=\; \mathcal{F} - \mathcal{F}\, P^{(++)}(\omega)\, \mathcal{F}^P(\omega)\,, \tag{3.60}$$

$$\tilde{R}^{\text{eff}}(\omega) \;=\; \tilde{A}(\omega) - \tilde{A}(\omega)\, \tilde{\mathcal{F}}(\omega)\, \tilde{R}^{\text{eff}}(\omega)\,, \tag{3.61}$$

with

$$\tilde{\mathcal{F}}(\omega) = Z^R(\omega)\, \mathcal{F}^P(\omega)\, Z^L(\omega) + \Phi(\omega)\,. \tag{3.62}$$

If the amplitude $\tilde{\mathcal{F}}(0)$ coincides with the interaction amplitude $\tilde{\mathcal{F}}$ satisfying Eq. (3.58), the renormalized response function $\tilde{R}^{\text{eff}}(\omega)$ has the pole at $\omega = 0$ corresponding to the spurious states by the same reasons as the QRPA response function $\tilde{R}(\omega)$. To ensure the fulfillment of the relationship $\tilde{\mathcal{F}}(0) = \tilde{\mathcal{F}}$ we use the fact that the interaction amplitude $\mathcal{F}$ entering Eq. (3.4) has not been constrained so far by any conditions besides the property of its energy independence. Let us now assume that the amplitude $\mathcal{F}$ satisfies the following equation:

$$\mathcal{F} = \mathcal{F}^P + \mathcal{F}^P P^{(++)}(0)\, \mathcal{F}\,, \tag{3.63}$$

where

$$\mathcal{F}^P = \left[Z^R(0)\right]^{-1}\left[\tilde{\mathcal{F}} - \Phi(0)\right]\left[Z^L(0)\right]^{-1}, \tag{3.64}$$

and the amplitude $\tilde{\mathcal{F}}$ is determined by Eq. (3.58). If the Eqs. (3.63) and (3.64) are fulfilled, then it follows from Eqs. (3.60) and (3.62) that $\tilde{\mathcal{F}}(0) = \tilde{\mathcal{F}}$. Consequently, both function $\tilde{R}^{\text{eff}}(\omega)$ and function $R^{\text{eff}}(\omega)$ have the poles at $\omega = 0$ corresponding to the spurious states. It means that these states are eliminated, at least energetically, within self-consistent version of the QTBA defined by the above equations.

## 4   CONCLUSIONS

In this paper the problem of the microscopic description of excited states of the even-even open-shell atomic nuclei is considered. The generalized Green function formalism (GGFF) has been presented and used to formulate the model including pairing, two-quasiparticle (2q), and more complex, first of all quasiparticle-phonon correlations. The GGFF is a modification of the existing versions of Green function formalism which is more suitable for solving the problem considered here. Within the GGFF the normal and the anomalous Green functions in the Fermi systems with pairing are treated in a unified way in terms of the components of generalized Green functions in a doubled space. This treatment is

analogous to the method used in Ref. [17]. In the GGFF this method is extended to the Fermi systems interacting through the two-, three-, and other many-particle effective forces, that is of importance for the nuclear physics where the many-particle forces play an essential role. Within the framework of this formalism the generalization of the model of Ref. [10] including the pairing correlations has been developed. The physical content of the model is determined by the quasiparticle time blocking approximation (QTBA) which allows one to keep the contributions of the 2q and 2q⊗phonon configurations, while excluding (blocking) more complicated intermediate states. It has been shown that within the QTBA the non-energy-weighted sum rule is fulfilled exactly. The model developed has been extended to include correlations in the quasiparticle pair entering 2q⊗phonon configuration, i.e. to include two-phonon configurations. Finally, the method of elimination of the spurious states within the self-consistent QTBA has been considered.

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